In conclusion, it is emphasized that the reduction procedure is always successful when applied to Ritz type difference equations; no claims are made about the relative merits of Ritz type difference equations compared to conventional equations in their accuracy in approximating the true solution.

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A Two Parameter Test Matrix

By P. J. Eberlein

1. Introduction. The $(N + 1) \times (N + 1)$ matrix A given by (1) below arose from a problem in the chemical theory of gases [1], the physically significant cases occurring when the parameter $s = 0, 1, 2, \cdots$. Since the eigenvalues[*] and eigenvectors of A are found explicitly, the matrix is of interest in itself as a test matrix for eigenvalue programs, especially for negative real s: when $s = -2, -3, \cdots -2N$, the matrix is defective with two or more pairs of eigenvectors coalescing; elsewhere in the range $-2N < s < -2$ at least one pair of eigenvectors is nearly parallel. In this range, the positive roots of the characteristic polynomial are ill-conditioned, especially for $s < -(N + 1)$.

The matrix, its eigenvalues, and its right and left eigenvectors are given in section 2; a few numerical experiments are described in section 3.

2. The Matrix. Let

$$(1) \quad A = (a_{ij}) = \begin{pmatrix} -N & N+s & 0 & 0 & \vdots \\ N & -(3N+s-2) & 2(N+s-1) & 0 & \\ 0 & 2(N-1) & -(5N+2s-8) & 3(N+s-2) & 0 \\ 0 & 0 & 3(N-2) & \cdot & \cdot \\ \cdots & & & & \end{pmatrix}$$

* Conjectured by Brauner and Wilson.

or generally

$$a_{ii} = -[(2i + 1)N + is - 2i^2]$$

$$a_{ii+1} = (i + 1)(N + s - i)$$

(2)

$$a_{ii-1} = i(N - i + 1)$$

$$a_{ij} = 0 \quad |i - j| > 1 \qquad\qquad i, j = 0, 1, 2 \cdots N$$

s is an arbitrary parameter, and $N + 1$ is the order of the matrix.

The eigenvalues of A are

(3) $$\lambda_j = -j(s + j + 1) \qquad\qquad j = 0, 1, 2 \cdots N.$$

Corresponding components of the left eigenvectors are

(4) $$x_i^{(j)} = \frac{1}{\binom{N}{i}} \sum_{k=0}^{q} (-1)^k \binom{N - k}{N - i}\binom{j}{k}\binom{s + j + k}{k}, \qquad i = 0, 1 \cdots N,$$

where $q = \min{(i, j)}$, or

(5) $$x_{N-m}^{(j)} = (-1)^j \frac{1}{\binom{N}{m}} \sum_{p=0}^{r} (-1)^p \binom{N - p}{m - p}\binom{j}{p}\binom{s + j + p}{j}$$

where $r = \min{(m, j)}$ and $m = N, N - 1, \cdots 2, 1, 0$.

Let n be an integer $1 \leqq n \leqq N$. The components of the right eigenvectors are

(6) $$y_i^{(j)} = \binom{N + s - i}{N - i} x_i^{(j)} \qquad unless\ s = -n\ and\ j \geqq n.$$

If $s = -n$ and $j \geqq n$

$$y_i^{(j)'} = \binom{N + s - i}{N - i}' x_i^{(j)} \qquad\qquad i \leqq N - n$$

(7)

$$y_i^{(j)'} = \binom{N + s - i}{N - i} x_i^{(j)'} \qquad\qquad i > N - n$$

where the prime indicates that a factor $(n + s)$ has been deleted.

The proofs are left as an exercise for the reader at the suggestion of the referee. We note also that

If $s = 2n$ $(1 \leqq n \leqq N)$ then $\lambda_j = \lambda_{2n-j-1}$ and $x_i^{(j)} = x_i^{(2n-j-1)}$ for $j = 0, 1,$ $2 \cdots n - 1$. If $s = -2n - 1$ $(1 \leqq n \leqq N - 1)$ then $\lambda_j = \lambda_{2n-j}$ and $x_i^{(j)} = x_i^{(2n-j)}$ for $j = 0, 1, 2 \cdots n$.

3. Numerical Experiments. A few numerical experiments for various N and s have been made using the real program described in [2] on the IBM 7070 and the CDC 1604. The following examples were typical; the negative roots were found with good accuracy and the approximations to the positive roots were usually

very bad. The non-defective cases, with several vectors nearly parallel, were sometimes worse than the defective cases.

$N = 6$ $s = -4$

actual roots		0,		0	2,	2	-1	-4	-10	-18
approximations (7070)		.0228	$-.0225$		1.995,	2.004	-1	-4	-10	-18

$N = 6$ $s = -6$

actual roots	-6	0, 0	4, 4	6, 6
approximations (7070)	-6	$-.0056 \pm .111i$	$3.969 \pm .363i$	$6.036 \pm .2215i$

$N = 6$ $s = -6.5$

actual roots	-3	0	2.5	4.5	6	7, 7.75
approximations (7070)	-3	.0066	2.436	5.103	5.151	$7.401 \pm .405i$

The effect of higher precision is seen in the following example which was run on both the 7070 (8 digits) and the 1604 (10 digits)

$N = 10$ $\kappa = -14$

actual roots	approximations (CDC 1604)	approximations (IBM 7070)
0	-2×10^{-7}	.0026
12	12.00004	11.898
22	21.995	20.195
30, 30	$29.638 \pm 1.134i$	$23.843 \pm 7.16i$
36, 36	$34.993 \pm 2.94i$	$33.926 \pm 12.92i$
40, 40	$40.024 \pm 3.39i$	36.977
42, 42	$43.347 \pm 1.49i$	$46.387 \pm 10.99i$
		52.594

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Multivariate Polynomial Approximation for Equidistant Data

By B. Mond

Abstract. The theory of polynomial approximation for evenly spaced points is extended to multivariate polynomial approximation. It is also shown how available tables prepared for univariate approximation can be used in the multivariate case.

1. Introduction. Assume $f(x)$ is given for $x = x_1, x_2, \cdots, x_n$ and it is desired to approximate $f(x)$ by a polynomial of degree p, $1 \le p < n$, i.e.

$$f(x) \approx \sum_{i=0}^{p} a_i x^i.$$